4.6

NON-HOMOGENEOUS LDE WITH VARIABLE COEFs OR FORCING TERMS NOT ONLY INVOLVE SUMS/PRODUCTS OF POLYNOMIALS, e^{kx}, $\cos bx$, $\sin bx$

CONSIDER $y'' + y = \sec x \longrightarrow r^2 + 1 = 0 \longrightarrow r = \pm i$

$$y_h = C_1 \cos x + C_2 \sin x$$

GUESS

$$y_p = v_1 \cos x + v_2 \sin x$$

WHERE $v_1 = v_1(x)$

AND $v_2 = v_2(x)$

~~UNDET COEF~~

$$\cancel{y_p = A_1 \sec x + A_2 \sec x \tan x + A_3 \sec x \tan^2 x + A_4 \sec^3 x + A_5 \sec^3 x \tan x + \ldots}$$

$\sec x(\sec^2 x - 1)$

$\sec^3 x - \sec x$

$$y_p' = v_1' \cos x - v_1 \sin x + v_2 \cos x + v_2' \sin x$$

$$= \boxed{v_1' \cos x + v_2' \sin x} - v_1 \sin x + v_2 \cos x$$

ADDITIONAL CONDITION

$$v_1' \cos x + v_2' \sin x = 0$$

SO $y_p' = -v_1 \sin x + v_2 \cos x$

$$y_p'' = -v_1' \sin x - v_1 \cos x + v_2' \cos x - v_2 \sin x$$

$$y_p'' + y_p = -v_1'\sin x - v_1\cos x + \cancel{v_1\cos x}$$
$$+ v_2'\cos x - \cancel{v_2\sin x + v_2\sin x}$$

$$= \text{①} \; -v_1'\sin x + v_2'\cos x = \sec x$$

ADDITIONAL CONDITION ② $v_1'\cos x + v_2'\sin x = 0$

2 EQNS
2 UNKNOWNS
v_1', v_2'

① * cos x $\quad -\cancel{v_1'\sin x\cos x} + v_2'\cos^2 x = 1$

② * sin x $\quad \cancel{v_1'\sin x\cos x} + v_2'\sin^2 x = 0$

$$v_2'(\cancel{\cos^2 x + \sin^2 x}) = 1$$
$$1$$

$$v_2' = 1 \longrightarrow v_2 = x + \cancel{C_2}$$

② $v_1'\cos x + 1 \cdot \sin x = 0$

$$v_1' = \frac{-\sin x}{\cos x} \longrightarrow v_1 = \ln|\cos x| + \cancel{C_1}$$

$u = \cos x$
$du = -\sin x\,dx$
$\int \frac{1}{u}\,du$

$$y_p = (\ln|\cos x| + \cancel{C_1})\cos x + (x + \cancel{C_2})\sin x$$
$$= \cos x\ln|\cos x| + \cancel{C_1\cos x} + x\sin x + \cancel{C_2\sin x}$$
$$y = \cos x\ln|\cos x| + x\sin x + C_1\cos x + C_2\sin x$$

SUPPOSE y_1, y_2 ARE L.I. SOL'NS OF $y'' + py' + qy = 0$

IE. $y_h = c_1 y_1 + c_2 y_2$

TO SOLVE $y'' + py' + qy = g$

LET $y_p = v_1 y_1 + v_2 y_2$

$$y_p' = \boxed{\cancel{v_1' y_1 + v_2' y_2}} + v_1 y_1' + v_2 y_2'$$

ADD'L COND

$$v_1' y_1 + v_2' y_2 = 0$$

$$y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''$$

$$y_p'' + p y_p' + q y_p = v_1' y_1' + v_2' y_2' \underbrace{\boxed{\cancel{+ v_1 y_1'' + p v_1 y_1' + q v_1 y_1}} \; \boxed{\cancel{+ v_2 y_2'' + p v_2 y_2' + q v_2 y_2}}}_{=0}$$

$$v_1 (y_1'' + p y_1' + q y_1) = v_1 \cdot 0 = 0$$

$$= \boxed{\begin{aligned} v_1' y_1' + v_2' y_2' &= g \\ v_1' y_1 + v_2' y_2 &= 0 \end{aligned}}$$

2 EQ'NS

2 UNKNOWNS v_1', v_2'

$$① \quad v_1' y_1' + v_2' y_2' = g$$
$$② \quad v_1' y_1 + v_2' y_2 = 0$$

$$① * -y_1 \quad \cancel{-v_1' y_1 y_1'} - v_2' y_1 y_2' = -g y_1$$
$$② * y_1' \quad \cancel{v_1' y_1 y_1'} + v_2' y_1' y_2 = 0$$
$$v_2' \underbrace{(y_1' y_2 - y_1 y_2')}_{-\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -(y_1 y_2' - y_1' y_2)} = -g y_1$$

$$-v_2' W[y_1, y_2] = -g y_1$$

$$v_2' = \frac{g y_1}{W[y_1, y_2]}$$

$$v_2 = \int \frac{g y_1}{W[y_1, y_2]} dx$$

ALSO $$v_1 = -\int \frac{g y_2}{W[y_1, y_2]} dx$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$\boxed{y_p = -y_1 \int \frac{g y_2}{W} dx + y_2 \int \frac{g y_1}{W} dx}$$

WHERE $W = W[y_1, y_2]$

RECONSIDER $y'' + y = \sec x$ g COEF OF y'' MUST BE 1 OR ELSE YOU USE WRONG g

$y_1 = \cos x \quad , \quad y_2 = \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x - {}^-\sin^2 x = 1$$

$$y_p = -y_1 \int \frac{g y_2}{W} dx + y_2 \int \frac{g y_1}{W} dx$$

$$= -\cos x \int \frac{\sec x \cdot \sin x}{1} dx$$

$$+ \sin x \int \frac{\sec x \cdot \cos x}{1} dx$$

$$= (-\cos x) \int \frac{\sin x}{\cos x} dx + (\sin x) \int dx$$

$u = \cos x \rightarrow du = -\sin x \rightarrow \int -\frac{1}{u} du = {}^-\ln|u| = {}^-\ln|\cos x|$

$$= (-\cos x) - \ln|\cos x| + (\sin x)\, x$$

$$= (\cos x) \ln|\cos x| + x \sin x$$

$$y'' - 2y' + y = \underbrace{x^{-1}e^{x}}_{g}$$

$$r^2 - 2r + 1 = 0$$

$$r = 1, 1$$

$$y_1 = e^x \qquad y_2 = xe^x$$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} + xe^{2x} - xe^{2x}$$

$$= e^{2x}$$

$$y_p = -y_1 \int \frac{g y_2}{W}\,dx + y_2 \int \frac{g y_1}{W}\,dx$$

$$= -e^x \int \frac{\cancel{x^{-1}}\cancel{e^x} \cdot \cancel{x}\cancel{e^x}}{\cancel{e^{2x}}}\,dx + xe^x \int \frac{x^{-1}\cancel{e^x} \cdot \cancel{e^x}}{\cancel{e^{2x}}}\,dx$$

$$= -e^x \int 1\,dx + xe^x \int x^{-1}\,dx$$

$$= \cancel{-xe^x} + xe^x \ln|x|$$

$$y_p = xe^x \ln|x|$$

$$y = xe^x \ln|x| + c_1 e^x + c_2 xe^x$$